the distance of two triangles and the gradient and hessian of its square

Bowen Ren

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1 Simplification

In the section, we are going to demonstrate that the minimum distance of two triangles is the minimum distance of the 9 edge-edge distance and 6 point-triangle distance. Consider two triangles which are not intersect.

Let's firstly consider a trivial case: if they are parallel, the minimum distance are equal to $\sqrt{dist_1^2 + dist_2^2}$, where $dist_1$ denotes the fixed distance of the two planes, and $dist_2$ denotes the minimum distance between the two triangles when we translate one of the original triangle along the perpendicular line between two planes to the plane of the other triangle. If the two triangles intersect after translation, then the minimum distance is exactly the distance of the two planes, which can be represent as one of the 9 edge-edge distance and 6 point-triangle distance. Otherwise, it is also obvious that the minimum distance of two triangles in the same plane which do not intersect is exactly one of the 9 edge-edge distance.

Afterwards, we assume that the two triangles are not parallel. Denote the two triangles as $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. Consider the projection of $\boldsymbol{\beta}$ on the plane of $\boldsymbol{\alpha}$. Suppose the distance between $\mathbf{A}$ in $\boldsymbol{\alpha}$ and $\mathbf{B}$ in $\boldsymbol{\beta}$ is the shortest. Suppose that $\mathbf{A}$ is in the interior of $\boldsymbol{\alpha}$, then $\mathbf{A}$ is the projection point of $\mathbf{B}$(or $\mathbf{A}$ is not the point in $\boldsymbol{\alpha}$ which has the shortest distance with $\mathbf{B}$). Consider the projection of $\boldsymbol{\alpha}$ on the plane of $\boldsymbol{\beta}$. Since $\mathbf{B}$ is the point in $\boldsymbol{\alpha}$ which has the shortest distance with $\mathbf{A}$, it is easy to demonstrate that $\mathbf{B}$ is either a vertex or a point on the edge while the edge is parallel to the plane of α. For the latter case, we can move point $\mathbf{A}$ parallel to this edge. In conclusion, we have proven the previous assertion: the minimum distance of two triangles is the minimum distance of the 9 edge-edge distance and 6 point-triangle distance.

2 Gradient and Hessian

As we proved above, we can divide the distance we previously calculated into two types: edge-edge distance and point-triangle distance.

2.1 point-triangle distance

Let the vertices of the triangle is $\mathbf{q_0}, \mathbf{q_1}, \mathbf{q_2}$, the point is $\mathbf{p_0}$, and the distance of $t_0 * \mathbf{q_0} + t_1 * \mathbf{q_1} + t_2 * \mathbf{q_2}$ and $\mathbf{p_0}$ is the shortest, where $t_0 + t_1 + t2 = 1$. It implies that

$$\frac{\partial t_0}{\partial x} + \frac{\partial t_1}{\partial x} + \frac{\partial t_2}{\partial x} = 0 \tag{1}$$

for any variable x. Let $\mathbf{d} = \mathbf{p0} - t_0 * \mathbf{q_0} - t_1 * \mathbf{q_1} - t_2 * \mathbf{q2}$. And noticed that d is perpendicular to $\mathbf{q_1} - \mathbf{q_0}$ and $\mathbf{q_2} - \mathbf{q_0}$. Hence we can solve $t_0, t_1, t_2$by the Matrix equation

$$\begin{pmatrix} \mathbf{q_0} \cdot (\mathbf{q_1} - \mathbf{q_0}) & \mathbf{q_1} \cdot (\mathbf{q_1} - \mathbf{q_0}) & \mathbf{q_2} \cdot (\mathbf{q_1} - \mathbf{q_0}) \\ \mathbf{q_0} \cdot (\mathbf{q_2} - \mathbf{q_0}) & \mathbf{q_1} \cdot (\mathbf{q_2} - \mathbf{q_0}) & \mathbf{q_2} \cdot (\mathbf{q_2} - \mathbf{q_0}) \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} t_0 \\ t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} \mathbf{p_0} \cdot (\mathbf{q_1} - \mathbf{q_0}) \\ \mathbf{p_0} \cdot (\mathbf{q_2} - \mathbf{q_0}) \\ 1 \end{pmatrix} \tag{2}$$

Therefore, We can calculate the partial gradient of t_0, t_1, t_2 to $\mathbf{q_0}, \mathbf{q_1}, \mathbf{q_2}, \mathbf{p_0}$ correspondingly. Then

$$\begin{aligned} \frac{\partial \mathbf{d}^2}{\partial \mathbf{q_0}} &= 2\mathbf{d} \left(-t_0 \cdot \mathbf{I} - \mathbf{q_0} \cdot \left(\frac{\partial t_0}{\partial \mathbf{q_0}} \right)^T - \mathbf{q_1} \cdot \left(\frac{\partial t_1}{\partial \mathbf{q_0}} \right)^T - \mathbf{q_2} \cdot \left(\frac{\partial t_2}{\partial \mathbf{q_0}} \right)^T \right) \\ &\overset{\text{use the equation 1}}{=} -2t_0\mathbf{d} - 2\mathbf{d}\left(\mathbf{q_1} - \mathbf{q_0}\right) \cdot \left(\frac{\partial t_1}{\partial \mathbf{q_0}} \right)^T - 2\mathbf{d}\left(\mathbf{q_2} - \mathbf{q_0}\right) \cdot \left(\frac{\partial t_2}{\partial \mathbf{q_0}} \right)^T \\ &= -2t_0\mathbf{d} \end{aligned}$$

The last equation is because $\mathbf{q_1} - \mathbf{q_0}, \mathbf{q_2} - \mathbf{q_0}$is perpendicular to d. Similarly,

$$\begin{aligned} \frac{\partial \mathbf{d}^2}{\partial \mathbf{q_1}} &= 2\mathbf{d} \left(-t_1 \cdot \mathbf{I} - \mathbf{q_0} \cdot \left(\frac{\partial t_0}{\partial \mathbf{q_1}} \right)^T - \mathbf{q_1} \cdot \left(\frac{\partial t_1}{\partial \mathbf{q_1}} \right)^T - \mathbf{q_2} \cdot \left(\frac{\partial t_2}{\partial \mathbf{q_1}} \right)^T \right) \\ &= -2t_2\mathbf{d} \end{aligned}$$

$$\begin{aligned} \frac{\partial \mathbf{d}^2}{\partial \mathbf{q_2}} &= 2\mathbf{d} \left(-t_2 \cdot \mathbf{I} - \mathbf{q_2} \cdot \left(\frac{\partial t_0}{\partial \mathbf{q_2}} \right)^T - \mathbf{q_1} \cdot \left(\frac{\partial t_1}{\partial \mathbf{q_2}} \right)^T - \mathbf{q_2} \cdot \left(\frac{\partial t_2}{\partial \mathbf{q_2}} \right)^T \right) \\ &= -2t_2\mathbf{d} \end{aligned}$$

$$\begin{aligned} \frac{\partial \mathbf{d}^2}{\partial \mathbf{p_0}} &= 2\mathbf{d} \left(\mathbf{I} - \mathbf{q_0} \cdot \left(\frac{\partial t_0}{\partial \mathbf{p_0}} \right)^T - \mathbf{q_1} \cdot \left(\frac{\partial t_1}{\partial \mathbf{p_0}} \right)^T - \mathbf{q_2} \cdot \left(\frac{\partial t_2}{\partial \mathbf{p_0}} \right)^T \right) \\ &= 2\mathbf{d} \end{aligned}$$

Finally,

$$\frac{\partial^2 \mathbf{d}^2}{\partial(\mathbf{q_0})^2} = \frac{\partial(-2t_0\mathbf{d})}{\partial \mathbf{q_0}}$$
$$= \frac{\partial(-2t_0)}{\partial \mathbf{q_0}}\mathbf{d} - 2t_0 * \frac{\partial \mathbf{d}}{\partial \mathbf{q_0}}$$

$$\frac{\partial^2 \mathbf{d}^2}{\partial(\mathbf{q_1})^2} = \frac{\partial(-2t_1)}{\partial \mathbf{q_1}}\mathbf{d} - 2t_1 * \frac{\partial \mathbf{d}}{\partial \mathbf{q_1}}$$

$$\frac{\partial^2 \mathbf{d}^2}{\partial(\mathbf{q_2})^2} = \frac{\partial(-2t_2)}{\partial \mathbf{q_2}}\mathbf{d} - 2t_2 * \frac{\partial \mathbf{d}}{\partial \mathbf{q_2}}$$

$$\frac{\partial^2 \mathbf{d}^2}{\partial(\mathbf{p_0})^2} = 2 * \frac{\partial \mathbf{d}}{\partial \mathbf{p_0}}$$

2.2 edge-edge distance

Let the vertices of these two edges be respectively $\mathbf{p_0}, \mathbf{p_1}; \mathbf{q_0}, \mathbf{q_1}$(3d vector). Suppose the minimum distance between the two edges is the distance of $\mathbf{p_0} + t_0 * (\mathbf{p_1} - \mathbf{p_0})$ and $\mathbf{q_0} + t_1 * (\mathbf{q_1} - \mathbf{q_0})$, where t_0, t_1 are in the interval [0,1]. Let $\mathbf{d} = \mathbf{p_0} + t_0 * (\mathbf{p_1} - \mathbf{p_0}) - \mathbf{q_0} + t_2 * (\mathbf{q_1} - \mathbf{q_0})$. And noticed that $\mathbf{d}$ is perpendicular to $\mathbf{q_1} - \mathbf{q_0}$ and $\mathbf{p_1} - \mathbf{p_0}$. Hence we can solve $t_0, t_1$by

$$\begin{pmatrix} -(\mathbf{p_1} - \mathbf{p_0}) \cdot (\mathbf{p_1} - \mathbf{p_0}) & (\mathbf{p_1} - \mathbf{p_0}) \cdot (\mathbf{q_1} - \mathbf{q_0}) \\ -(\mathbf{q_1} - \mathbf{q_0}) \cdot (\mathbf{p_1} - \mathbf{p_0}) & (\mathbf{q_1} - \mathbf{q_0}) \cdot (\mathbf{q_1} - \mathbf{q_0}) \end{pmatrix} \begin{pmatrix} t_0 \\ t_1 \end{pmatrix} = \begin{pmatrix} (\mathbf{p_0} - \mathbf{q_0}) \cdot (\mathbf{p_1} - \mathbf{p_0}) \\ (\mathbf{p_0} - \mathbf{q_0}) \cdot (\mathbf{q_1} - \mathbf{q_0}) \end{pmatrix} \tag{3}$$

Therefore, We can calculate the partial gradient of t_0, t_1 to $\mathbf{q_0}, \mathbf{q_1}, \mathbf{p_0}, \mathbf{p_1}$ correspondingly. Then

$$\frac{\partial \mathbf{d}^2}{\partial \mathbf{p_0}} = 2\mathbf{d}\left((1 - t_0) \cdot \mathbf{I} + (\mathbf{p_1} - \mathbf{p_0}) \cdot \left(\frac{\partial t_0}{\partial \mathbf{p_0}}\right)^T - (\mathbf{q_1} - \mathbf{q_0}) \cdot \left(\frac{\partial t_1}{\partial \mathbf{p_0}}\right)^T\right)$$
$$= 2(1 - t_0)\mathbf{d}$$

The last equation is because that $\mathbf{q_1} - \mathbf{q_0}, \mathbf{p_1} - \mathbf{p_0}$is perpendicular to $\mathbf{d}$. Similarly,

$$\frac{\partial \mathbf{d}^2}{\partial \mathbf{p_1}} = 2\mathbf{d}\left(t_0 \cdot \mathbf{I} + (\mathbf{p_1} - \mathbf{p_0}) \cdot \left(\frac{\partial t_0}{\partial \mathbf{p_1}}\right)^T - (\mathbf{q_1} - \mathbf{q_0}) \cdot \left(\frac{\partial t_1}{\partial \mathbf{p_1}}\right)^T\right)$$
$$= 2t_0\mathbf{d}$$

$$\frac{\partial \mathbf{d}^2}{\partial \mathbf{q_0}} = 2\mathbf{d}\left((t_1 - 1) \cdot \mathbf{I} + (\mathbf{p_1} - \mathbf{q_0}) \cdot \left(\frac{\partial t_0}{\partial \mathbf{q_0}}\right)^T - (\mathbf{q_1} - \mathbf{q_0}) \cdot \left(\frac{\partial t_1}{\partial \mathbf{q_0}}\right)^T\right)$$
$$= 2(t_1 - 1)\mathbf{d}$$

$$
\begin{aligned}
\frac{\partial \mathbf{d}^2}{\partial \mathbf{q_1}} &= 2\mathbf{d}\left(-t_1 \cdot \mathbf{I} + (\mathbf{p_1} - \mathbf{p_0}) \cdot \left(\frac{\partial t_0}{\partial \mathbf{q_1}}\right)^T - (\mathbf{q_1} - \mathbf{q_0}) \cdot \left(\frac{\partial t_1}{\partial \mathbf{q_1}}\right)^T\right) \\
&= -2t_1\mathbf{d}
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\frac{\partial^2 \mathbf{d}^2}{\partial (\mathbf{p_0})^2} &= \frac{\partial (2(1-t_0)\mathbf{d})}{\partial \mathbf{p_0}} \\
&= \frac{\partial (2(1-t_0))}{\partial \mathbf{p_0}}\mathbf{d} + 2(1-t_0) * \frac{\partial \mathbf{d}}{\partial \mathbf{p_0}}
\end{aligned}
$$

$$
\frac{\partial^2 \mathbf{d}^2}{\partial (\mathbf{p_1})^2} = \frac{\partial (2t_0)}{\partial \mathbf{p_1}}\mathbf{d} + 2t_0 * \frac{\partial \mathbf{d}}{\partial \mathbf{p_1}}
$$

$$
\frac{\partial^2 \mathbf{d}^2}{\partial (\mathbf{q_0})^2} = \frac{\partial (2(t_1-1))}{\partial \mathbf{q_0}}\mathbf{d} + 2(t_1-1) * \frac{\partial \mathbf{d}}{\partial \mathbf{q_0}}
$$

$$
\frac{\partial^2 \mathbf{d}^2}{\partial (\mathbf{q_1})^2} = \frac{\partial (-2t_1)}{\partial \mathbf{q_1}}\mathbf{d} - 2t_1 * \frac{\partial \mathbf{d}}{\partial \mathbf{q_1}}
$$